

WHY THE CROSS PRODUCT IS A VECTOR
AND THE DOT PRODUCT IS A SCALAR
by Len Fellman (lenfellman@comcast.net)

The geometric definition of the dot product and cross product of two three-dimensional vectors \mathbf{v} and \mathbf{w} making a positive angle θ (less than 180°) are easy to state if we only care about their magnitudes:

$$\left. \begin{aligned} |\mathbf{v} \cdot \mathbf{w}| &= \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta \\ \|\mathbf{v} \times \mathbf{w}\| &= \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta \end{aligned} \right\} (1)$$

Why don't we define both $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ using these two scalar expressions and let it go at that?

To answer this question, we will examine the behavior of these two expressions under vector addition.

In order for a product of vectors to be useful, it must be *bilinear*:

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w}$$

and

$$(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{w} = \mathbf{v}_1 \times \mathbf{w} + \mathbf{v}_2 \times \mathbf{w}.$$

It should also exhibit some type of commutativity:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \mathbf{w} \cdot \mathbf{v} \\ \mathbf{v} \times \mathbf{w} &= -\mathbf{w} \times \mathbf{v}. \end{aligned}$$

We write $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ and $\theta_1 = \angle(\mathbf{v}_1, \mathbf{w})$, $\theta_2 = \angle(\mathbf{v}_2, \mathbf{w})$, $\theta_3 = \angle(\mathbf{v}_3, \mathbf{w})$.

Also, for any vector \mathbf{v} we will write v for $\|\mathbf{v}\|$.

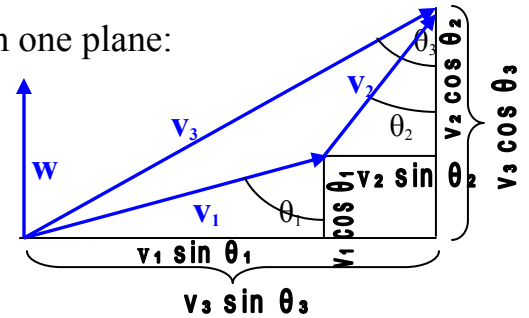
We will also simplify matters by taking \mathbf{w} to have length 1, which is possible because $\|\mathbf{w}\|$ factors out of equations (1):

$$\begin{aligned} |\mathbf{v} \cdot \mathbf{w}| &= v \cos \theta \\ \|\mathbf{v} \times \mathbf{w}\| &= v \sin \theta \end{aligned}$$

We want to compare $v_1 \cos \theta_1 + v_2 \cos \theta_2$ with $v_3 \cos \theta_3$,
and to compare $v_1 \sin \theta_1 + v_2 \sin \theta_2$ with $v_3 \sin \theta_3$.

First look at the picture when \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{w} are all in one plane:

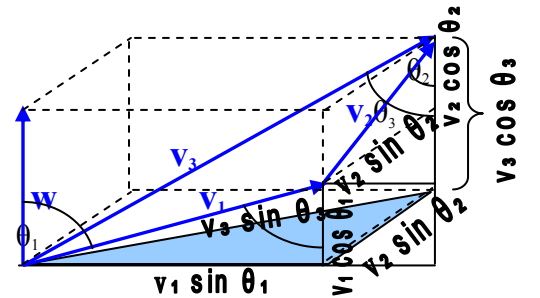
Observe that the three expressions involving cosine are all parallel, as are the three expressions involving sine. This means that the expressions $v \cos \theta$ and $v \sin \theta$ “add like scalars”, so we could define $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ as scalars in this case.



Note in passing that there is an analogue of the cross-product in two dimensions, taking $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$ to the signed quantity $x_1 y_2 - x_2 y_1$. This function is bilinear and anti-commutative. It is independent of the coordinate system since it can be defined geometrically as $vw \sin \theta$. It can also be thought of as $\mathbf{v} \cdot \mathbf{w}^\perp$, where \mathbf{w}^\perp means the vector \mathbf{w} rotated 90° counterclockwise. It can of course be associated with a vector perpendicular to \mathbf{v} and \mathbf{w} , giving the ordinary cross-product.

Now let’s look at the picture when \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{w} are in different planes:

Here \mathbf{v}_1 is on the front face of the box, \mathbf{v}_2 is on the right face, and \mathbf{v}_3 forms the diagonal of the box. Each of the three vectors is part of a right triangle, with legs $v_i \cos \theta_i$ and $v_i \sin \theta_i$ as shown.



The legs involving *cosine* are all in the same direction as \mathbf{w} (vertical); hence the expressions $v \cos \theta$ add like *scalars*. However, the legs involving *sine* are *not* in the same direction: they form a *triangle* (the shaded region on the base of the box): so these expressions add like *vectors*.

However, these segments are not in the proper direction to define the vectors $\mathbf{v}_i \times \mathbf{w}$. They are each perpendicular to \mathbf{w} , but they aren’t perpendicular to \mathbf{v}_i ($i = 1, 2$, or 3). (They measure “the component of $\mathbf{v}_i \times \mathbf{w}$ perpendicular to \mathbf{w} ”.) If we used these segments to define $\mathbf{v}_i \times \mathbf{w}$, the result would be a “very non-commutative” operation. We need $\mathbf{v}_i \times \mathbf{w}$ to be perpendicular to \mathbf{v}_i as well as to \mathbf{w} .

To achieve this, we *rotate the shaded triangle* 90° in the horizontal (base) plane, as shown here: The resulting vectors \mathbf{u}_i are still perpendicular to \mathbf{w} , but now each \mathbf{u}_i is also perpendicular to \mathbf{v}_i as well. We arrive at the vector cross-product.

